The analytical solutions obtained in the manner described here make it possible to analyze the entire multiplicity of frontal regimes in the multicomponent ($n \geq 3$) dynamics of adsorption ($c_{0m} > c_m^0$) and desorption ($c_{0m} < c_m^0$) for various values of the concentration c_{0m}, c_m^0 ($1 \leq m \leq n$).

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A SET OF STEADY-STATE SOLUTIONS OF THE EVOLUTION EQUATION FOR PERTURBATIONS IN ACTIVE-DISSIPATIVE MEDIA

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Steady-state periodic solutions have been calculated numerically. It is demonstrated that an even set of such solutions comes about as a consequence of a successive cascade of bifurcations.

In recent times, researchers into the wave processes in nonconservative media have expressed great interest in an equation of the form

$$H_t + 4HH_x + H_{xx} + H_{xxxx} = 0. \tag{1}$$

This interest is generated by the fact that in terms of form it is one of the simplest nonlinear equations which could possibly be imagined, so that with its appearance in the simulation of the nonlinear behavior of perturbations for a rather large class of active-dissipative media it functions for the latter in as extensive a role as the well-known KdV equation for conservative media.

Thus, in the description of the waves at the surface of a liquid film flowing freely down an inclined plane, such an equation has been derived in [1, 2], for the counterflow motion of a film and a gas we find the derivation of such an equation in [3], and for the perturbations at the boundaries separating two viscous liquids in a horizontal channel, the derivation of the equation is to be found in [4].

Linear stability analysis demonstrated that the trivial solution H = 0 of Eq. (1) is unstable relative to perturbations of the form $\exp[i\alpha(x - ct)]$ with wave numbers $\alpha < 1$ (perturbations with $\alpha > 1$ are attenuated). The growth of such perturbations over time can be curtailed through the action of nonlinear effects, as a result of which steady-state nonlinear regimes are formed.

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It follows from linear stability theory that the periodic solution of infinitely small amplitude branches from the trivial solution of Eq. (1) when $\alpha = 1$. In its vicinity, the solution of small but finite amplitude can be achieved analytically in the form of a series over the small parameter, which is frequently represented by the amplitude itself. It is extended into the region of linear instability ($\alpha < 1$), i.e., a soft type of branching.

For the steady-state traveling wave $H(\xi)$ ($\xi = x - ct$) Eq. (1) is written in the form

$$-cH_{\xi} + 4HH_{\xi} + H_{\xi\xi} + H_{\xi\xi\xi\xi} = 0. \tag{2}$$

In finding the periodic solutions of Eq. (2) with a wavelength $\lambda = 2\pi/\alpha$, in light of the fact that it is invariant relative to the transformations

$$H \to -H,\ \xi \to -\xi,\ c \to -c,\ H \to H + \mathrm{const},\ c \to c + 4\ \mathrm{const}, \tag{3}$$

we will limit ourselves to an examination only of those for which $c \geq 0$, $\int\limits_0^{\lambda} H\, d\xi = 0$. Thus, we arrive at the boundary-value problem in which the eigenvalues are the phase velocity c of the wave, and where the parameter is the wave number α.

The periodic solutions of Eq. (2) with noticeable amplitude are found numerically. For this, they are presented in the form of a Fourier series

$$H = \sum_{n=-\infty}^{\infty} H_n \exp[i\alpha n\xi]. \tag{4}$$

Since H is a real function, then $H_{-n} = \bar{H}_n$ (the bar indicates the complex conjugacy operation). Leaving the first N harmonics in Eq. (4), we will substitute it into (2). Having equated the coefficients for identical exponents to zero, we obtain a system of N complex equations for the real unknown c and for N complex $H_1, \ldots, H_N$:

$$(-i\alpha cn - \alpha^2 n^2 + \alpha^4 n^4) H_n + 2i\alpha n \sum_{m=n-N}^{N} H_m H_{n-m} = 0, \quad n = 1, \ldots, N. \tag{5}$$

In view of the invariance of Eq. (2) relative to the displacement of the coordinate

$$\xi \to \xi + \mathrm{const} \tag{6}$$

the origin can be set so that, for example,

$$\mathrm{Re}(H_1) = 0. \tag{7}$$

With consideration of (7), system (5) is defined. We made use of the Newton method in the numerical solution of this system. In concluding series (4) we chose the number of harmonics so as to fulfill the relationship $|H_N| / \sup |H_n| \leq 10^{-3}$. For this, the number N in the calculations had to be changed in dependence on α in limits from 10 to 40.

The basic difficulty in the solution of Eq. (2) by this method involves the specification of the initial approximation, sufficiently close to the solution. For the first set of solutions, branching from the trivial with $\alpha = 1$, we used the analytical solution. Movement along the parameter α is achieved in continuous fashion, i.e., the wave-number interval is chosen so that in using the initial approximation of the earlier-found solution it will enter the region of convergence.

It has been determined in [5] that the first set of solutions can be continuously extended in the direction of smaller wave numbers to values of $\alpha_* = 0.4979$. At this point each odd harmonic of series (4) is equated to zero. As a result, we have a solution with a wave number $\alpha = 2\alpha_* = 0.9958$, and here it develops that it coincides with the earlier-derived α. Thus, this set of solutions closes on itself. Let us note that for all α out of the region of existence of solutions there exists for this family of solutions a phase velocity c = 0. It follows from (3) that such solutions are antisymmetric. In the present study we will limit ourselves to the examination of those solutions for which c = 0. With this purpose in mind, we will use a regular procedure to undertake bifurcational analysis of branching from the first set of periodic steady-state solutions of Eq. (1). We demonstrate how an even set of such solutions appears. A portion of the set has been found in [6, 7]. Let us in choosing the coordinate origin for ξ so as to satisfy (7), make the real parts of all of the harmonics in these solutions equal to zero. The procedure for the solution of the problem is covered in detail in [8], and we will therefore cover it here only in brief.

Let $H_0(\xi)$ be a periodic solution of Eq. (2) with the wave number α. To investigate its stability with regard to infinitely small perturbations, we will substitute $H = H_0(\xi) + h(\xi, t)$ into (1) and we will linearize it with respect to $h(\xi, t)$. We will then have a linear equation for h with periodic coefficients for ξ of the same period as H_0. When we take into consideration that the variable t is not included here explicitly, and in accordance with the Floquet theorem dealing with the solutions of linear equations with periodic coefficients, the solution of this equation, limited for all ξ, is presentable in the following form [1]:

$$h = \exp\,[-\gamma t + i\alpha Q\xi]\varphi(\xi) + \text{c.c.}, \tag{8}$$

where φ is a periodic function of the same period as $H_0(\xi)$; Q is a real parameter. After simple transformations we obtain [1]

$$A\varphi + B\varphi' + (1 - 6\alpha^2Q^2)\varphi'' + 4i\alpha Q\varphi''' + \varphi^{IV} = \gamma\varphi \tag{9}$$

($A = 4H_0' + 4i\alpha QH_0 - \alpha^2Q^2 + \alpha^4Q^4 - i\alpha Qc$, $B = 4H_0 + 2i\alpha Q - 4i\alpha^3Q^3 - c$, with the prime denoting differentiation with respect to ξ).

Thus, the study of the stability of periodic steady-state solutions for Eq. (1) reduces to a study (with various values of Q) of the spectrum of eigenvalues of γ for which (9) exhibits a periodic solution of the same period as H_0. The solution is stable if for any Q for all γ $\mathrm{Re}(\gamma) \geq 0$. It becomes clear from (8) that we can restrict ourselves to an examination of Q at any individual interval, for example, at [−0.5 to 0.5]. Carrying out the complex conjugacy operation in (9), it is easy to prove that it is enough to examine the solution of (9) for the case in which $0 \leq Q \leq 0.5$.

Boundary-value problem (9) was solved numerically. For this purpose, subjecting (9) to Fourier transformation, we obtain an infinite system of linear algebraic uniform equations for φ_n. Equating to zero all φ_n with $n \geq N$, we come to the final approximation

$$[-i\alpha c(Q+n) - \alpha^2(Q+n)^2 + \alpha^4(Q+n)^4]\varphi_n + 4i\alpha(Q+n)\Sigma H_m\varphi_{n-m} = \gamma\varphi_n.$$

It follows from (8) that if at some point (α, Q) the real portion of any eigenvalue vanishes, then from the original a new wave regime branches out. With $\mathrm{Im}(\gamma) = 0$ it is possible to have the generation of nonsteady regimes. New steady-state solutions arise from the solution of H_0 if

$$\gamma(\alpha, Q) = 0. \tag{10}$$

With Q irrational, a double-period regime is generated. If $Q = P/r$ is a rational number, a regime periodic with respect to ξ is formed with a new wave number $\alpha_{new} = \alpha/r$. Expression (8) is used to find new solutions for Eq. (2) for wave numbers from the vicinity of α_{new} as the initial approximation. The function h in expression (8) is made proportional to the eigenfunction φ satisfying Eq. (9) with the eigenvalue of γ from (10). The calculations show that the continuous motion in the vicinity of these points is most conveniently achieved through utilization of any of the harmonics as the varying parameter. Subsequently, by changing to the parameter α, we also find a solution for α, distant from the points of origination for this set of α_{new}.

Some of the results from our study of the stability of the first set of solutions can be seen in Fig. 1, taken from [8, 9]. The cross-hatched area in the (α, Q) plane shows the region of stability. At its lower boundary, only the real portion of one of the eigenvalues passes through zero. The corresponding nonsteady regimes are generated from this line, i.e., we are dealing here with a bifurcation of the Landau–Hopf type. We can see that in agreement with the results from [1] the regimes with wave numbers belonging to the interval (0.837-0.778) are stable with respect to all infinitely small plane perturbations. On the curves 1-4 for any of the eigenvalues we find satisfaction of relationship (10). Subsequently, when speaking of new solutions branching away from these curves, we will refer to these as generating curves. With the aid of the latter we can easily find the wave numbers α_{new} with which new steady-state solutions are generated. The calculations demonstrate that these solutions are intricately interlaced with each other.

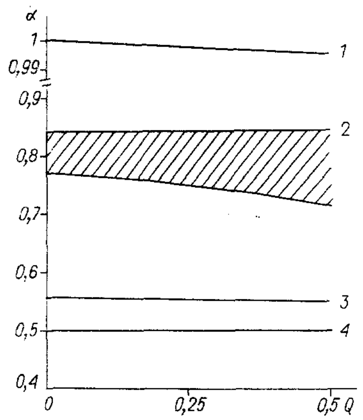

Fig. 1

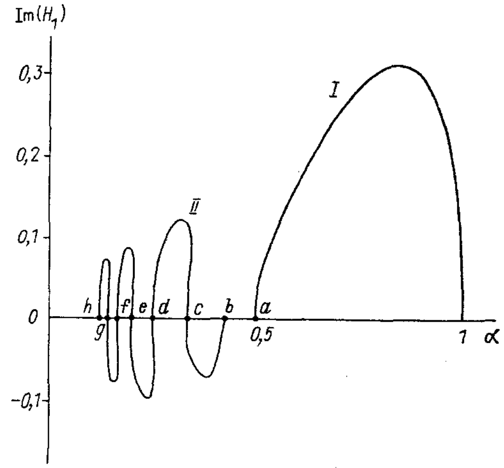

Fig. 2

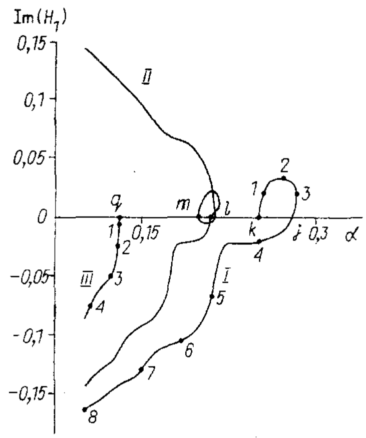

Fig. 3

Figure 2 provides a demonstration of some of the interrelated transitions of the new solutions from one to the other, and it shows the amplitude of the first harmonic $Im(H_1)$ as a function of α. Since for Eq. (2) Eq. (3) is valid, it becomes clear that all of the curves in Figs. 2 and 3 have their own mirror images relative to the $Im(H_1) = 0$ axis, and these reflections correspond to the same solution, shifted through a half period (these latter are not shown, so as not to make the figures overly cumbersome).

In view of (6), all of the solutions can be taken from $Im(H_1) \geq 0$, but the representation employed here allows us most clearly to see how the solutions make the transition from one to the other. It is precisely the point at which $Im(H_1) = 0$ in Figs. 2 and 3, with the exception of those cases stipulated, that serve as the points of generation for new solutions. If these points are associated with the branching from Q = 1/2, then a new solution is generated at these points (see, for example, points *a* and b, Fig. 2), and if this occurs from Q = P/r, r ≥ 3, then two new solutions are generated.

Curve I in Fig. 2 demonstrates how the first set disappears, closing on that set generated from the first maximum possible new wave number, i.e., the point *a* ($\alpha_a = 0.4979$). Indeed, at this point we have a solution which branches away from curve 1 (see Fig. 1) with Q = 1/2, α = 0.9958. In the following, in describing new sets of solutions, generated directly from the first set as per curves 1-4, speaking of the corresponding generating curve, we will not stress each time that we are dealing with the curves in Fig. 1.

Curve II (Fig. 2) shows that the new sets of solutions can make the transition from one to the other. Its boundary point b is connected to the branch from curve 2 with Q = 1/2. Moving continuously along this solution, we pass successively through the points at which new sets of solutions are generated, these associated with the branching from curve 1 with Q = 1/3, 1/4, 1/5, 1/6, 1/7, and 1/8 (points c, d, e, f, g, and h). On the basis of these results there arises the temptation to reveal that the new solutions, bifurcating from curve

1 with Q = 1/r, with a reduction in α, change to a solution which branches from this same curve with Q = 1/(r + 1).

A similar assertion is encountered in [6]. There, utilizing the general theory of the branching of solutions for nonlinear equations, we find the actual study of certain bifurcations from curves 1 and 2 with Q = 1/r and semianalytically the branching points a, b, c, and d have been determined (including j; see Fig. 3). The wave numbers $\alpha_a = 0.4979$, $\alpha_b = 0.4352$, $\alpha_c = 0.3323$, $\alpha_d = 0.2494$, $\alpha_f = 0.2923$ obtained in [6] are sufficiently close to those found with our method: $\alpha_a = 0.4979$, $\alpha_b = 0.4211$, $\alpha_c = 0.3323$, $\alpha_d = 0.2494$, $\alpha_j = 0.2803$. The development of the solutions was traced in [6] to $\alpha \approx 0.25$ and on the basis of calculations carried out the conclusion was drawn that the solution for Eq. (2) with the period $2\pi r$, bifurcating from the solution of the first set with period 2π, makes the transition to a solution with the period $2\pi(r + 1)$. However, our studies of branching from curves 2-4 show that this assertion does not represent the general situation.

Thus, in the branching from curve 2 with Q = 1/3 one of the solutions is continuous monotonically along α all the way to the minimum values of the wave numbers ($\alpha_{min} = 0.1$). The second solution branches in the direction of large α, and then quickly turns around and merges with the solution continued by curve 4 with Q = 1/2. These three sets are shown in Fig. 3 by line I (the corresponding wave profiles can be found in Fig. 4). Here, and in Figs. 4 and 5 the numbers on the profiles are uniquely associated with the numbered points on the corresponding curves in Fig. 3.

Let us note that at the instant of generation the wave numbers of the sets generated by curve 1 with Q = 1/4 and curve 4 with Q = 1/2 are very close to each other: $\alpha_d = 0.2494$ and $\alpha_k = 0.2505$, respectively. Nevertheless, this method of analyzing the bifurcations distinguishes between them completely.

The set of solutions arising from curve 2 with Q = 1/4 evolved with a change in α, similar to the sets of solutions which are generated from this same curve with Q = 1/3. Indeed, one of these is extended from the point of generation of these sets, i.e., $\alpha_\ell = 0.2099$, in the direction of smaller α and exists all the way to α_{min}. The second set branches in the direction of larger α, then turns into the region of smaller α and can be extended to $\alpha_m = 0.1992$. As demonstrated by analysis, this is the point of origination for two new sets from curve 1 with Q = 2/5. Thus we again have the merging of two different sets of solutions.

The uniqueness of the point α_m in comparison with similar points considered above lies in the fact that in accordance with (8) in its vicinity it is the second harmonic $\mathrm{Im}(H_2)$ that is decisive, rather than the first harmonic, as in the case of branching with Q = 1/r.

The second set of solutions arising at the point α_m also continues into the region of small α, at least to α_{min}. In the $[\mathrm{Im}(H_1), \alpha]$ plane, this solution initially enters the region of large α, intersects the axis of abscissas at the point $\alpha_+ = 0.211$, and then departs

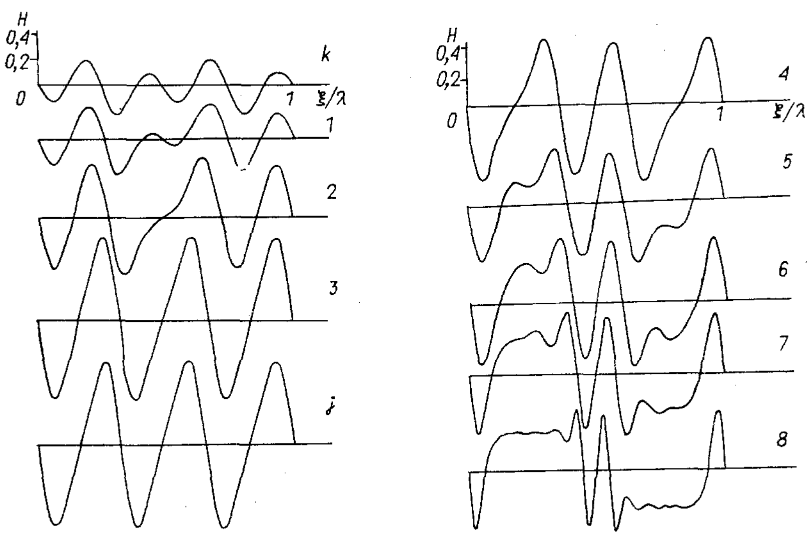

Fig. 4

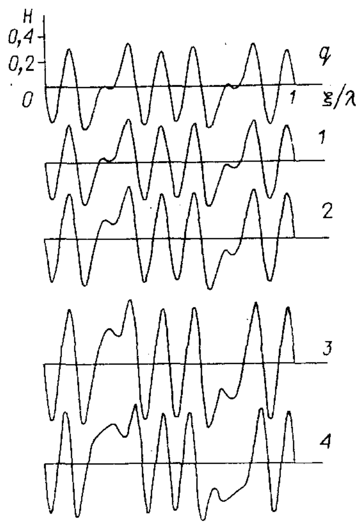

Fig. 5

monotonically into the region of small α. Unlike all of the other points lying at the intersection of the curves (see Figs. 2 and 3) with this axis, point α_+ is not connected to the generation of any new solutions. At this point only the single harmonic $\mathrm{Im}(H_1)$ vanishes. In particular, the imaginary part of the second harmonic $\mathrm{Im}(H_2)$ on passage of this point in the indicated direction monotonically increases in absolute magnitude. Line II in Fig. 3 represents the set of solutions generated at the points α_ℓ and α_m.

For all of the sets of periodic solutions of Eq. (2) that we have examined here, the values of the phase velocity satisfy c = 0. It can easily be demonstrated that as $\alpha \to 0$ they cannot become soliton solutions. Therefore, these sets either exist in the finite interval of wave numbers, closing on each other, or they are extended to all small α. With a reduction in α the wave period shows an ever-increasing number of local maxima and minima.

We can see from the cited examples of branching from curves 1-4 in Fig. 1 how with a reduction in α increasingly new solutions arise. In this case, the distances between the points of generation for new regimes are reduced and at the limit $\alpha \to 0$ we have an even set of solutions. In turn, studying the stability of these solutions, on the (α, Q) plane we can obtain new generating curves (10), analogous to curves 1-4 in Fig. 1. It is clear that in principle this procedure can be carried out an unlimited number of times.

As an illustration of the above, Fig. 5 shows a set of solutions which arises as a result of such "secondary" branching. This regime of branching from the set of solutions, shown partially in Fig. 3 by curve I, lies above the axis of abscissas. Let us remember that it arises in the branching from curve 4 with Q = 1/2. The branching of the new regime occurs from the point α = 0.2602, also with Q = 1/2. The corresponding amplitude of the first harmonic is shown by line III in Fig. 3.

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